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PLANE AND AXISYMMETRIC CONTACT PROBLEMS FOR ROUGH ELASTIC BODIES

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Plane and axisymmetric contact problems for a rough layer are studied in a nonlinear formulation. In the particular case of a very thick layer, the principal term of the kernel under consideration in the nonlinear integral equation agrees with the kernel examined in [1, 2]. The solution of the problems reduces to investigating nonlinear integral equations of Hammerstein type, for which we use successive approximation. Numerical results are presented which show the change in the nature of the pressure distribution under the stamp when roughness of the foundation is taken into account.

Contact problems of the theory of elasticity for rough bodies were considered in a linear formulation in [3 —5]. Shtaerman first obtained the equation of the plane contact problem for an elastic rough body on the basis of an assumption about the proportionality of the additional local displacements because of the spreading of the roughness in the contact zone by the normal pressure. However, as results of a number of experimental studies show [6, 7], the closure of the rough bodies making contact is proportional to the pressure to the power α $(\alpha \leqslant 1)$ because of the deformation of the microprojections. In such a formulation some plane contact problems were examined in [1, 2, 8]. An approximate solution of the axisymmetric problem is presented in [9].

1. Let us consider the plane contact problem for an elastic rough strip $|x| < \infty$, $0 < y < h$. A rigid stamp, the shape of whose surface is given by the equation $y = g(x)$, $(g(0) = 0)$, is impressed on the upper boundary of the strip by a force P. Outside the contact section $(-a, a)$, the upper boundary of the strip is not loaded. Two cases of strip support are investigated in parallel:

1) The strip lies on a rigid foundation without friction; in this case the boundary conditions have the form (δ is the settlement of the stamp)

$$\begin{aligned} &\tau_{xy}(x, 0) = 0, \ v(x, 0) = 0, \ |x| < \infty \\ &\tau_{xy}(x, h) = 0, \ \sigma_y(x, h) = 0, \ a < |x| < \infty \\ &\tau_{xy}(x, h) = 0, \ v(x, h) = g(x) + \delta, \ |x| < a \end{aligned} \tag{1.1}$$

2) The strip is fixed rigidly along the foundation; then the conditions on the line of discontinuity $y = 0$ change, taking the form

$$u(x, 0) = 0, \ v(x, 0) = 0$$

Let us consider the normal displacements $v(x, h)$ of the boundary of the elastic strip to be comprised of the displacement v_1 due to the strain of the microprojections defined as follows:

$$v_1 = A\,[p\,(x)]^{\alpha},\ A > 0 \tag{1.2}$$

Here $p\,(x)$ is the contact pressure distribution function, A is a coefficient characterizing the deformation properties of the rough layer, and α is an exponent found on the basis of a reference surface curve $(\alpha \leqslant 1)$. Moreover, elastic displacements v_2 of the strip occur, which are determined on the basis of the boundary conditions according to [10]

$$v_2 = \frac{2\,(1-\nu^2)}{\pi E}\int_{-a}^{a} k\left(\frac{\xi - x}{h}\right) p\,(\xi)\,d\xi \tag{1.3}$$

The function $k\,(t)$ has the form

$$k\,(t) = \int_0^{\infty} \frac{L\,(u)}{u}\cos ut\,du \tag{1.4}$$

The specific form of the function $L\,(u)$ depends on the boundary conditions.

For problem 1

$$L\,(u) = \frac{\mathrm{ch}\,2u - 1}{\mathrm{sh}\,2u + 2u} \tag{1.5}$$

For problem 2

$$L\,(u) = \frac{2\varkappa\,\mathrm{sh}\,2u - 4u}{2\varkappa\,\mathrm{ch}\,2u + 4u^2 + 1 + \varkappa^2},\quad \varkappa = 3 - 4\nu \tag{1.6}$$

Let us write the condition for contact between the stamp and the strip and the equilibrium condition in dimensionless coordinates. To do this, we introduce the notation

$$x_1 = \frac{x + a}{2a},\quad \eta = \frac{\delta}{2a},\quad \lambda = \frac{2a}{h} \tag{1.7}$$

$$g_1\,(x_1) = \frac{1}{2a}\,g\,[a\,(2x_1 - 1)],\quad A_1 = \frac{A}{2a}\left[\frac{2\,(1-\nu^2)}{\pi E}\right]^{-\alpha}$$

$$p_1\,(x_1) = \frac{2\,(1-\nu^2)}{\pi E}\,p\,[a\,(2x_1 - 1)],\quad P_1 = \frac{1-\nu^2}{\pi E a}\,P$$

We then obtain

$$\int_0^1 k\,[\lambda\,(t - x_1)]\,p_1\,(t)\,dt + A_1\,[p_1\,(x_1)]^{\alpha} = g_1\,(x_1) + \eta \tag{1.8}$$

$$P_1 = \int_0^1 p_1\,(x_1)\,dx_1 \tag{1.9}$$

Therefore, the solution of the problem formulated reduces to solving the nonlinear integral equation (1.8) under the condition (1.9), whereupon the pressure under the stamp and the settlement of the stamp are determined.

Equation (1.8) is an equation of Hammerstein type. Let us reduce it to canonical form. To do this we introduce the new function

$$\psi\,(x_1) = A_1\,[p_1\,(x_1)]^{\alpha} - g_1\,(x_1) - \eta \tag{1.10}$$

Then

$$A_1^{-1/\alpha}\int_0^1 k\,[\lambda\,(t - x_1)]\,[\psi\,(t) + g_1\,(t) + \eta]^{1/\alpha}dt + \psi\,(x_1) = 0 \tag{1.11}$$

$$P_1 = A_1^{-1/\alpha}\int_0^1 [\psi\,(x_1) + g_1\,(x_1) + \eta]^{1/\alpha}\,dx_1$$

Successive approximations can be used to solve the equation (1. 11) of Hammerstein type. For instance, setting $\psi_0(x_1) \equiv 0$ and successively

$$\psi_{n+1}(x_1) = -A_1^{-1/\alpha}\int_0^1 k[\lambda(t-x_1)][\psi_n(t)+g_1(t)+\eta]^{1/\alpha}dt$$

Let us prove compliance with the sufficient conditions for convergence of this method for (1. 11).

1) The kernel $k(t)$ of the integral equation (1. 11) can be represented in the following form by taking (1. 4) — (1. 6) into account

$$k(t) = -\ln|t| + F(t), \quad 0 \leqslant |t| < \infty$$

where $F(t)$ is a continuous function. The kernel $k(t)$ evidently belongs to the class L_2.

2) The function $f(t, u) = A_1^{-1/\alpha}[u + g_1(t) + \eta]^{1/\alpha}$ uniformly satisfies a Lipschitz condition of the form

$$|f(t, u_1) - f(t, u_2)| < c(t)|u_1 - u_2| \tag{1.12}$$

in the range of variation of the function $u = \psi(x_1)$. In fact, since the normal pressure $p_1(x_1)$ is non-negative, then

$$-g_1(x_1) - \eta \leqslant u \leqslant 0$$

follows from (1. 10) and the properties of the kernel in (1. 11).

The derivative $\partial f(t, u)/\partial u = \alpha^{-1}A_1^{-1/\alpha}[u + g_1(t) + \eta]^{1/\alpha-1}$ is bounded by the value $\alpha^{-1}A_1^{-1/\alpha}[g_1(t) + \eta]^{1/\alpha-1}$ $(0 < \alpha \leqslant 1)$ in the range of variation of the argument u . Hence, the Lipschitz condition (1. 12) is satisfied in which

$$c(t) = \frac{1}{\alpha}A_1^{-1/\alpha}[g_1(t) + \eta]^{1/\alpha-1}$$

3) The function $f(t, 0) = A_1^{-1/\alpha}[g_1(t) + \eta]^{1/\alpha}$, evidently belongs to the class L_2.

Upon compliance with conditions 1) — 3) the method of successive approximations converges [11] if only the parameters of the problem satisfy the inequality

$$\alpha^{-2}A_1^{-2/\alpha}\int_0^1 [g_1(x)+\eta]^{2/\alpha-2}\left\{\int_0^1 k^2[\lambda(t-x)]\,dt\right\}dx < 1 \tag{1.13}$$

Upon compliance with condition (1. 13), the sequence of functions $\{\psi_n(x_1)\}$ has a limit which will indeed be the unique solution of (1. 11).

Let us prove that this solution is unique. Let us assume the oposite, i. e., that two solutions of (1. 11) exist: $\psi_1(x_1)$ and $\psi_2(x_1)$. Then a nontrivial solution of the equation

$$\Psi(x_1) + A_1^{-1/\alpha}\int_0^1 k[\lambda(t-x_1)]\,\varphi[t, \Psi(t)]\,dt = 0 \tag{1.14}$$

$(\Psi(x_1) = \psi_1(x_1) - \psi_2(x_1),\ \varphi[t, \Psi(t)] = [\psi_2(t) + \Psi(t) + g_1(t) + \eta]^{1/\alpha} - [\psi_2(t) + g_1(t) + \eta]^{1/\alpha})$

should exist.

Let us multiply (1. 14) by the function $\varphi[x_1, \Psi(x_1)]$ and let us integrate over the segment $(0, 1)$. We obtain

$$A_1^{-1/\alpha}\int_0^1\int_0^1 k[\lambda(t-x_1)]\,\varphi[t, \Psi(t)]\,\varphi[x_1\Psi(x_1)]\,dt\,dx_1 = \tag{1.15}$$

$$-\int_0^1 \Psi(x_1)\,\varphi[x_1, \Psi(x_1)]\,dx_1$$

The function $\varphi[x_1, \Psi(x_1)]$ is positive for $\Psi(x_1) > 0$ and negative for $\Psi(x_1) < 0$. Therefore,

$$\int_0^1 \Psi(x_1)\,\varphi[x_1, \Psi(x_1)]\,dx_1 > 0, \quad \Psi(x_1) \not\equiv 0$$

The kernel $k[\lambda(t-x)]$ is non-negative, i. e., the inequality

$$J(\omega) = \int_0^1\int_0^1 k[\lambda(t-x)]\,\omega(t)\,\omega(x)\,dt\,dx \geqslant 0$$

holds for any continuous function $\omega(x)$ not identically zero in the range $(0,1)$. Indeed, the functional $J(\omega)$ can be represented in the form

$$J(\omega) = \int_0^1 \omega(x)\left\{\int_0^1 k[\lambda(t-x)]\,\omega(t)\,dt\right\}dx$$

The expression in the braces is, as follows from (1. 3), the displacement to the accuracy of a positive constant, which the boundary of the strip will undergo on the contact area under the effect of the distributed load $\omega(x)$. Therefore, the functional $J(\omega)$ is, to the accuracy of a positive factor, the total work produced by the arbitrary pressures $\omega(x)$ on the appropriate displacements of points of the contact area, which is always non-negative. By virtue of the non-negativity of the kernel, the left side of (1. 15) is non-negative. Therefore, (1. 5) is valid if and only if $\Psi(x_1) \equiv 0$.

By knowing the solution of (1. 11) as the limit of the sequence of functions $\psi_n(x)$, the pressure in dimensionless coordinates can be found by means of (1. 10). The settlement η of the stamp is found from (1. 9).

Let us note that the pressure cannot be infinite at the ends of the contact area. Indeed, by assuming that the pressure has an integrable power singularity of the form $x_1^{-\theta}$ $(0 < \theta < 1)$ at the point $x_1 = 0$, and taking into account that the kernel of the integral equation (1. 8) has a singularity of the form $\ln x_1$, we obtain that the left side in (1. 8) has a singularity of the order of $x_1^{-\alpha\theta}$, while there is no singularity in the right side, which proves the assertion mentioned above.

In the case of a smooth stamp making contact with a rough elastic layer an additional condition $p(-a) = p(a) = 0$ expressing the continuity of the function for the pressure on the boundary of the elastic layer, exists to determine the unknown boundaries of the contact area $(-a, a)$.

2. Different plane contact problems for a rough layer can be solved by the method

elucidated in Sect.1 by determining the nature of the pressure distribution on the boundary of the rough layer as a function of the layer thickness, the roughness parameters, the elastic characteristics of the layer, etc.

As an illustration of a contact problem, let us consider the frictionless impression of a rigid stamp with a flat base $g(x) = 0$ in a thick rough layer. In this case, we have the integral equation (1.8) and the condition (1.9) in which $g_1(x_1) = 0$ to determine the pressure and the kernel is representable in the form

$$k(t) = -\ln|t| + a_0$$

where $a_0 = -0.352$ for the first boundary value problem, and $a_0 = -0.527$ for the second boundary value problem. Such an asymptotic representation of the kernel (1.4) is valid for sufficiently thick strips when $\lambda^2 = [2a/h]^2 = o(1)$ [10].

By solving (1.8) by successive approximations under these conditions, we obtain for the pressure on the contact area

$$p_1(x_1) = A_1^{-1/\alpha}[\psi(x_1) + \eta]^{1/\alpha}$$

where $\psi(x_1)$ is the limit of the sequence of functions $\{\psi_n(x_1)\}$ and

$$\psi_{n+1}(x_1) = A_1^{-1/\alpha}\int_0^1 [\ln|t - x_1| + c_0][\psi_n(t) + \eta]^{1/\alpha}\,dt$$

$$c_0 = \ln(2a/h) - a_0$$

This limit exists if condition (1.13) is satisfied, which takes the following form in this case

$$\alpha^{-2}A_1^{-2/\alpha}\eta^{2/\alpha-2}(c_0^2 - 3c_0 + 3.5) < 1$$

On the basis of results of experimental investigations [6, 7], the following values of the dimensionless parameters were taken for the numerical computations: $\alpha = 0.4$, $A_1 = 1$, $c_0 = -3$. Graphs of the pressure distribution in the case of the dimensionless loads $P_1^{(1)} = 0.6\cdot10^{-2}$ and $P_1^{(2)} = 0.75\cdot10^{-2}$ acting on the stamp are represented by curves *1* and *2*, respectively, in Fig. 1. The values of P_1 are related to the true load values by the last formula in (1.7). The depths corresponding to the loads $P_1^{(1)}$ and $P_1^{(2)}$ are $\eta^{(1)} = 0.15$, $\eta^{(2)} = 0.17$, where $\eta = \delta/(2a)$. As is seen from the graph, the pressure increases under the effect of the larger load, especially at the ends of the contact area. Under the effect of identical $0.41\cdot10^{-2}$ loads, the depth of the stamp and the pressure under it will vary, as computations showed, as a function of the smoothness of the surface treatment, which is characterized by the dimensionless parameters α and A_1. For $\alpha = 0.4$ and $A_1 = 0.75$ the stamp penetrates to a depth $\eta = 0.1$, while for $\alpha = 0.4$ and $A_1 = 0.35$ (smoother treatment), the stamp penetrates a smaller amount $\eta = 0.06$. Graphs of the pressure distribution in these two cases are shown by curves *3* and *4* in Fig. 1, respectively; a graph of the pressure distribution without taking account of the roughness is presented by the dashed line. The computations exhibited satisfactory convergence of the method of successive approximations and its effectiveness. To 10^{-4} accuracy it turned out to be sufficient to calculate 15 – 20 approximations of the function $\psi(x)$. The process converges for practically all reasonable values of the roughness parameters and the elastic characteristics of the material.

3. Let us examine the axisymmetric problem of impressing a circular stamp in a rough elastic half-space $(y < 0)$. The shape of the stamp surface making contact is given by the equation $y = g(\rho)$ $(g(0) = 0)$. A force P presses the stamp into the half-space. The contact area is in the shape of a circle of radius a. The normal displacements of the elastic half-space boundary in the contact region, defined by the shape of the stamp and its settlement δ, are comprised of elastic displacements at points of the half-space boundary v_2, which are determined according to [10] on the basis of the boundary conditions

$$v_2 = \frac{1-\nu^2}{\pi E}\int_0^a\int_0^{2\pi}\frac{p(r)\,rd\,rd\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos\varphi}}$$

($p(r)$ is the distribution function of the contact pressures), and the displacements v_1 due to deformation of the microprojections, which are determined by (1.2).

Let us write the condition for contact between the stamp and the half-space boundary and the equilibrium condition in dimensionless coordinates in canonical form. To do this we introduce the notation

$$r_1 = \frac{r}{a},\quad p_1(r_1) = \frac{1-\nu^2}{\pi E}p(r_1 a),\quad g_1(r_1) = g\frac{(r_1 a)}{a}$$

$$\eta = \frac{\delta}{a},\quad A_1 = \frac{A}{a}\left[\frac{\pi E}{1-\nu^2}\right]^\alpha,\quad P_1 = \frac{1-\nu^2}{\pi E a^2}P$$

$$\psi(\rho_1) = A_1[p_1(\rho_1)]^\alpha - g_1(\rho_1) - \eta$$

We then obtain

$$A_1^{-1/\alpha}\int_0^1 k(r_1,\rho_1)\,r_1[\psi(r_1)+g_1(r_1)+\eta]^{1/\alpha}dr_1 + \psi(\rho_1) = 0 \tag{3.1}$$

$$k(r_1,\rho_1) = \int_0^{2\pi}\frac{d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}}$$

$$P_1 = 2\pi\int_0^1 p_1(r_1)\,r_1\,dr_1$$

As in Sect. 1, we will solve the equation of Hammerstein type (the first equation in (3.1)) by successive approximations. The kernel $k(r_1, \rho_1)$ of the integral equation (3.1) evidently belongs to the class L_2. It can also be confirmed that the function

$$f(r,u) = A_1^{-1/\alpha}r[u+g_1(r)+\eta]^{1/\alpha},\quad u \leqslant 0$$

uniformly satisfies the Lipschitz condition

$$|f(r, u_1) - f(r, u_2)| < c(r)\,|u_1 - u_2|$$

$$c(r) = \alpha A_1^{-1/\alpha}r[g_1(r)+\eta]^{1/\alpha-1}$$

and that the function $f(r, 0) = A_1^{-1/\alpha}r[g_1(r)+\eta]^{1/\alpha}$ belongs to the class L_2. Hence, upon compliance with the inequality

$$\alpha^{-2}A_1^{-2/\alpha}\int_0^1\int_0^1 \rho^2\,[g_1(\rho)+\eta]^{2/\alpha-2}k^2(r,\rho)\,dr\,d\rho<1$$

the sequence of functions $\{\psi_n(\rho_1)\}$ will converge almost everywhere to the solution of (3.1) [11]. The proof of the uniqueness of the solution is carried out analogously to the proof executed in Sect. 1.

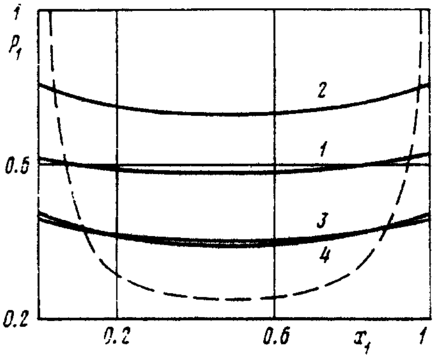

Fig. 1

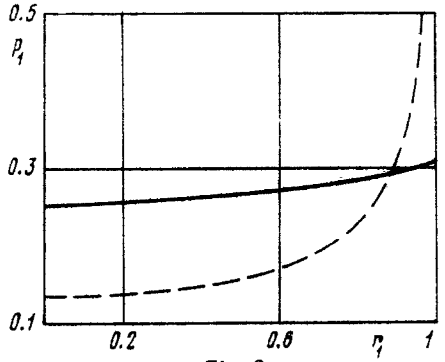

Fig. 2

As an illustration, let us consider the numerical solution of a problem on impressing a circular cylindrical stamp with a flat base $g(\rho)=0$ into a rough elastic half-space. The following numerical values were taken for the dimensionless parameters: $\alpha=0.4$, $A_1=0.9$, $\eta=0.1$. To 10^{-5} accuracy in the solution it turns out to be sufficient to evaluate 12 approximations of the function $\psi(x)$. The graph obtained for the function $p_1(r_1)$ representing the dimensionless pressure is presented in Fig. 2. The graph of the pressure distribution under the stamp is given by the dashed line when roughness of the base is not taken into account. In both cases a load of $P=0.8625\cdot 10^{-2}$ acts on the stamp.

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